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Cosmic Microwave Background (Cont'd):

As discussed, the dynamics of subhorizon perturbations can be described by that of a damped harmonic oscillator in an external field. The initial conditions for perturbations are set by primordial values. One should therefore start with a given mode when it is outside the horizon (i.e., "superhorizon" regime) and follow its evolution then.

Sachs-Wolfe Effect:

As mentioned before, one can choose various hypersurfaces to describe the evolution of a perturbed FRW universe. A very convenient choice is the "Newtonia" gauge, in which the metric is given by:

$$ds^2 = (1+2\Phi)dt^2 - a^2_{(t)}(1-2\Psi)\left[dr^2 + f^2_{(r)}\, d\Omega^2\right]$$

$$f_{(r)} = \begin{cases} \sinh r & \text{open} \\ r & \text{flat} \\ \sin r & \text{closed} \end{cases}$$

In the absence of stress, one can show that $\Phi = \Psi$, which results in:

$$ds^2 = (1+2\Phi)\,dt^2 - a^2_{(t)}\,(1-2\Phi)\left[dr^2 + f^2_{(r)}\,\sin^2 d\Omega\right] \qquad (I)$$

In a matter-dominated or radiation-dominated phase, it can be shown that $\dot{\Phi} = 0$. Therefore, Φ is a function of spatial Coordinates only. At superhorizon scales, one can divide the hypersurface into causally disconnected universes that have their own scale factor $a_{(t)}(1-2\Phi)^{1/2}$ and clock $dt\,(1+2\Phi)^{1/2}$. These universes evolve independently. The differences are due to different values of Φ (which is the gravitational potential in the Newtonian limit), which is related to the density perturbation $\delta\rho$. The Hubble rate for these universes follows:

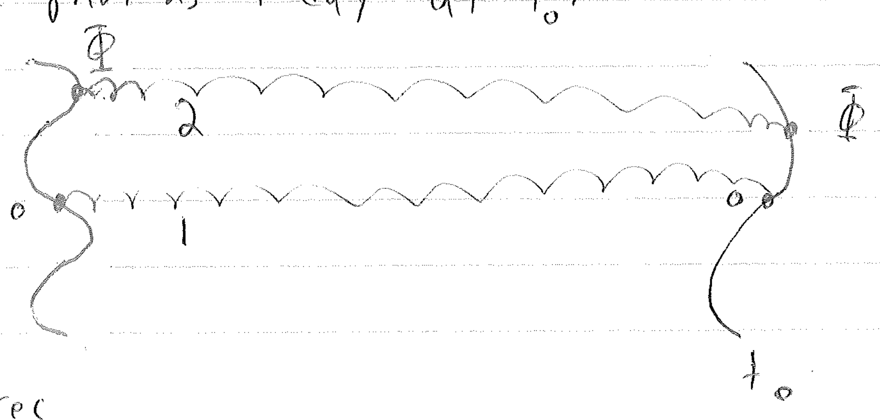
$$H^2_{(\Phi)} = \left(\frac{(1-2\Phi)^{1/2}\,da_{(t)}}{(1+2\Phi)^{1/2}\,dt}\right)^2 \left(\frac{1}{a_{(t)}(1-2\Phi)^{1/2}}\right)^2 = \frac{8\pi G}{3}\,(\rho + \delta\rho)$$

Considering that $\left(\frac{da}{dt}\right)^2 \frac{1}{a^2} = \frac{8\pi G\rho}{3}$, and in the linear regime $|\Phi| \ll 1$, we find:

$$\Phi = -\frac{1}{2}\frac{\delta\rho}{\rho} \qquad (\mathrm{II})$$

This relation is valid in both matter-dominated and radiation-dominated epochs.

Density perturbation $\delta\rho$ that induce a gravitational potential Φ imply that photons arriving from regions with different values of Φ undergo different amounts of gravitational redshift. To see this, consider a photon that leaves the surface of last scattering where $\Phi=0$ and another one that leaves the surface of last scattering where the gravitational potential is Φ. This happens at the time of recombination t_{rec}, and we observe these photons today at t_0:



Following what we did for an exact FRW universe, and taking into account dependence of the clock rate on Φ, we find for photon 1:

$$\frac{T_{1,0}}{T_{1,rec}} = \frac{a(t_{rec})}{a(t_0)}$$

For photon 2, on the other hand, we have:

$$\frac{T_{2,0}}{T_{2,rec}} = \frac{a(t_{rec})}{a(t_0)}(1+\Phi)$$

Comparing the temperatures of the two photons as observed today, we find:

$$\left(\frac{T_{2,0}}{T_{1,0}}\right)\left(\frac{T_{1,rec}}{T_{2,rec}}\right) = 1+\Phi$$

The gravitational redshift due to density perturbations is then given by:

$$z_{SW} = \Phi$$

This is the Sachs-Wolfe effect, which describes the difference in gravitational redshift due to perturbations.

An important point to note that there is an intrinstic temperature difference between the two photons since they are coming from regions with different energy densities, see Eq.(II) in above. The intrinsic difference is due to the fact that n_γ follows n_B because of the photon-baryon coupling;

adiabatic perturbations

$$\left(\frac{\delta T}{T}\right)_{int} = \frac{1}{3}\frac{\delta n_\gamma}{n_\gamma} = \frac{1}{3}\frac{\delta n_B}{n_B} = \frac{1}{3}\frac{\delta \rho_B}{\rho_B} \overset{\uparrow}{=} \frac{1}{3}\frac{\delta \rho}{\rho}$$

The intrinsic temperature fluctuations plus the Sachs-Wolfe effect result in observed temperature fluctuations as follows:

$$\underbrace{\left(\frac{\delta T}{T}\right)_{obs}}_{\text{observed}} = \underbrace{\left(\frac{\delta T}{T}\right)_{int}}_{\text{intrinsic}} + \underbrace{\Phi}_{\text{Sachs-Wolfe}}$$

In the matter-dominated phase, we have $\Phi = -\frac{1}{2}\frac{\delta \rho}{\rho}$. Thus:

$$\left(\frac{\delta T}{T}\right)_{obs} = -\frac{2}{3}\Phi + \Phi$$

$$\left(\delta T\right)_{obs} = \frac{1}{3}\Phi \qquad (\text{III})$$

This relation denotes the observed temperature fluctuations in terms of the gravitational potential on the surface of last scattering. We can rewrite it as follows:

$$\left(\frac{\delta T}{T}\right)_{obs} = \frac{-1}{6}\frac{\delta \rho}{\rho} = -\frac{1}{2}\left(\frac{\delta T}{T}\right)_{int} \qquad (\mathrm{IV})$$

This tells us that gravitational redshift dominates over intrinsic temperature fluctuations. As a result, photons arriving from an overdense region ($\left(\frac{\delta T}{T}\right)_{int} > 0$) have a lower temperature today ($\left(\frac{\delta T}{T}\right)_{obs} < 0$). Therefore, as long as superhorizon perturbations are concerned, cold spots in the CMB map represent overdense regions at the time of recombination, while hot spots are representative of the underdense regions.

One point to keep in mind is that expressions in Eqs. (III, IV) will be different in the radiation-dominated phase $\left(\frac{\delta T}{T}\right)_{obs} = \frac{1}{2}\Phi = -\left(\frac{\delta T}{T}\right)_{int}$